

Section 7.3. Trigonometric Substitution

A motivating example: prove that the area of a circle of radius R is πR^2 .

Solution: the upper semi-circle of radius R centered at $(0,0)$ has the equation $y = +\sqrt{R^2 - x^2}$. Therefore the area of the circle of radius R centered at $(0,0)$ is $A = 2 \int_{-R}^R \sqrt{R^2 - x^2} dx$, or $A = 4 \int_0^R \sqrt{R^2 - x^2} dx$.

This integral is difficult to compute by "traditional" methods.

The trick: make the substitution $x = R \sin \theta$, $dx = R \cos \theta d\theta$.

Thus $x=0$ corresponds to $\theta=0$, and $x=R$ corresponds to $\theta=\frac{\pi}{2}$.

Moreover, for $0 \leq \theta \leq \frac{\pi}{2}$, we have $0 \leq \sin \theta \leq 1$, and thus $0 \leq R \sin \theta \leq R$, whence $0 \leq x \leq R$, as needed.

$$\text{We therefore compute } A = 4 \int_0^R \sqrt{R^2 - x^2} dx = 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2 - R^2 \sin^2 \theta} R \cos \theta d\theta$$

Using $1 - \sin^2 \theta = \cos^2 \theta$, we get

$$\begin{aligned} A &= 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2(1 - \sin^2 \theta)} R \cos \theta d\theta = 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2 \cos^2 \theta} R \cos \theta d\theta \\ &= 4 \int_0^{\frac{\pi}{2}} R |\cos \theta| \cdot R \cos \theta d\theta. \end{aligned}$$

Note that for values of θ in $[0, \frac{\pi}{2}]$, $\cos \theta$ is non-negative $\Rightarrow |\cos \theta| = \cos \theta$.

$$\begin{aligned} \text{Therefore } A &= 4 \int_0^{\frac{\pi}{2}} R \cos \theta \cdot R \cos \theta d\theta = 4R^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = 4R^2 \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta \\ &= 4R^2 \left(\frac{1}{2}\theta + \frac{1}{4} \sin 4\theta \right) \Big|_0^{\frac{\pi}{2}} = 4R^2 \cdot \frac{1}{2} \frac{\pi}{2} = \boxed{\pi R^2} \quad ■ \end{aligned}$$

This type of substitution is called "inverse substitution":

- in U-substitution, we choose $u = \text{some function of } x$;
- in inverse substitution, we choose $x = \text{some function of } \theta$.

Since in this context we make use of trig. functions, we call this type of inverse substitution "trigonometric substitution".

Here's a table of common trig. substitutions:

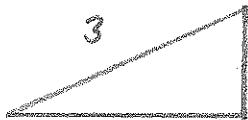
| Expression | substitution | domain of θ | Identity to use |
|--------------------|---------------------|---|-------------------------------------|
| $\sqrt{a^2 - x^2}$ | $x = a \sin \theta$ | $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ | $1 - \sin^2 \theta = \cos^2 \theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a \tan \theta$ | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ | $1 + \tan^2 \theta = \sec^2 \theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a \sec \theta$ | $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$ | $\sec^2 \theta - 1 = \tan^2 \theta$ |

Why do we need a domain for θ ? To avoid integrating absolute values.

Examples ① $\int \frac{x^2}{\sqrt{9-x^2}} dx$. let $x = 3 \sin \theta$, $dx = 3 \cos \theta d\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned} \text{Then we have } \int \frac{x^2}{\sqrt{9-x^2}} dx &= \int \frac{9 \sin^2 \theta}{\sqrt{9-9 \sin^2 \theta}} 3 \cos \theta d\theta = \int \frac{9 \sin^2 \theta}{3 \sqrt{1-\sin^2 \theta}} \cdot 3 \cos \theta d\theta \\ &= \int \frac{9 \sin^2 \theta}{3 \sqrt{\cos^2 \theta}} 3 \cos \theta d\theta = \int \frac{9 \sin^2 \theta}{3 \cancel{\cos \theta}} \cdot \cancel{3 \cos \theta} d\theta = 9 \int \sin^2 \theta d\theta \\ &= 9 \int \frac{1}{2} - \frac{1}{2} \cos 2\theta d\theta = 9 \left(\frac{1}{2}\theta - \frac{1}{4} \sin 2\theta \right) + C = \frac{9}{2}\theta - \frac{9}{4} \cdot 2 \sin \theta \cos \theta + C \end{aligned}$$

Here we have used $\sin 2\theta = 2 \sin \theta \cos \theta$. Now, $x = 3 \sin \theta \Rightarrow \sin \theta = \frac{x}{3}$



Therefore $\cos \theta = \frac{\sqrt{9-x^2}}{3}$, and $\theta = \sin^{-1}(\frac{x}{3})$. This gives us

$$\int \frac{x^2}{\sqrt{9-x^2}} dx = \frac{9}{2} \sin^{-1} \left(\frac{x}{3} \right) - \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} + C.$$

$$\textcircled{2} \quad \int \frac{dx}{x^4 \sqrt{9x^2-1}} = \int \frac{dx}{3x^4 \sqrt{x^2 - \frac{1}{9}}} \quad \text{let } x = \frac{1}{3} \sec \theta, dx = \frac{1}{3} \sec \theta \tan \theta d\theta;$$

$$\frac{\int \frac{1}{3} \sec \theta \tan \theta d\theta}{3 \cdot \frac{1}{3^4} \sec^4 \theta \cdot \sqrt{\frac{1}{9} (\sec^2 \theta - 1)}} = 3^3 \int \frac{\sec \theta \tan \theta d\theta}{\sec^4 \theta \sqrt{\tan^2 \theta}} = 27 \int \sec^3 \theta d\theta$$

$$= 27 \int \cos^3 \theta \, d\theta = 27 \int (\cos \cdot (1 - \sin^2 \theta)) \, d\theta \quad \text{let } u = \sin \theta, \, du = \cos \theta \, d\theta$$

$$= 27 \int 1 - u^2 \, du = 27 \left(u - \frac{u^3}{3} \right) + C = 27 \left(\sin \theta - \frac{\sin^3 \theta}{3} \right) + C.$$

Now, $x = \frac{1}{3} \sec \theta \Rightarrow \sec \theta = 3x \Rightarrow \cos \theta = \frac{1}{3x}$



So, $\sin \theta = \frac{\sqrt{9x^2 - 1}}{3x}$; with this, we finally have

$$\int \frac{dx}{x^4 \sqrt{9x^2 - 1}} = 27 \left(\frac{\sqrt{9x^2 - 1}}{3x} - \frac{1}{3} \left(\frac{\sqrt{9x^2 - 1}}{3x} \right)^3 \right) + C.$$

③ Evaluate $\int_1^e \frac{dy}{y \sqrt{\ln^2 y + 25}}$. Let $u = \ln y$, then $du = \frac{1}{y} dy$: Thus

$$\int \frac{dy}{y \sqrt{\ln^2 y + 25}} = \int \frac{du}{\sqrt{u^2 + 25}}. \quad \text{Now do a trig-sub with } u = 5 \tan \theta.$$

④ Evaluate $\int \frac{dx}{\sqrt{x^2 - 8x + 12}}$. Complete the square: $x^2 - 8x + 12 = (x-4)^2 - 4$.

Then $\int \frac{dx}{\sqrt{x^2 - 8x + 12}} = \int \frac{dx}{\sqrt{(x-4)^2 - 4}}$. let $u = x-4$, $du = dx$. Then we have

$$\int \frac{dx}{\sqrt{(x-4)^2 - 4}} = \int \frac{du}{\sqrt{u^2 - 4}}. \quad \text{Now do a trig-sub with } u = 2 \sec \theta.$$

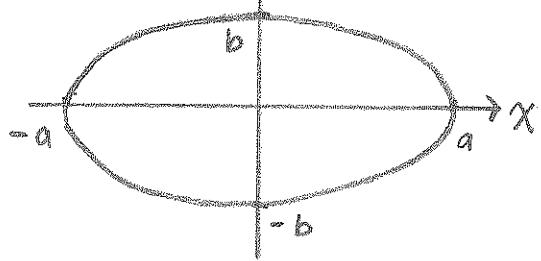
⑤ Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The top semi-ellipse has the equation $y = \sqrt{b^2 \left(1 - \frac{x^2}{a^2}\right)} = \frac{b}{a} \sqrt{a^2 - x^2}$.

The area of the ellipse is therefore

$$A = 2 \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} \, dx = \frac{4b}{a} * \text{Area of quarter-circle of radius } a$$

$$= \frac{4b}{a} * \frac{\pi a^2}{4} = \boxed{\pi ab}$$



⑥ Complete the square on $4x - x^2$

$$4x - x^2 = -(x^2 - 4x) = -[(x-2)^2 - 4] = 4 - (x-2)^2.$$